The Snake Lemma

Theorem *Consider a commutative diagram of R-modules of the form*

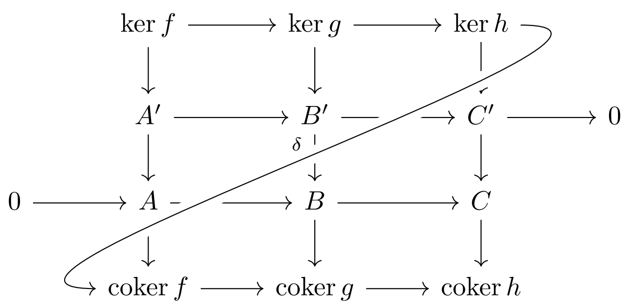
$$\begin{array}{ccccccccc} & & A' & \xrightarrow{\alpha_1} & B' & \xrightarrow{\beta_1} & C' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow[\alpha_2]{} & B & \xrightarrow[\beta_2]{} & C & & \end{array}$$

If the rows are exact, then there is an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h),$$

where $\delta : \ker(h) \to \operatorname{coker}(f)$ is the map $\delta(c') = {\alpha_2}^{-1} g {\beta_1}^{-1}(c')$. Furthermore, if $A' \to B'$ is monic, then so is $\ker(f) \to \ker(g)$, and if $B \to C$ is epi, then so is $\operatorname{coker}(g) \to \operatorname{coker}(h)$.

Proof. The snake picture to have in mind is as follows:

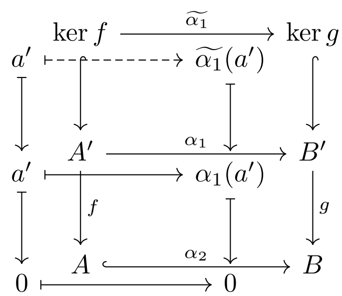


Notice that the columns create long exact sequences:

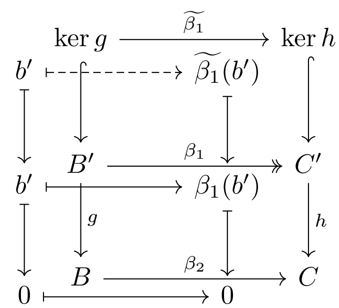
$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ & & \ker f & \xrightarrow{\widetilde{\alpha_1}} & \ker g & \xrightarrow{\widetilde{\beta_1}} & \ker h & & \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ & & A' & \xrightarrow{\alpha_1} & B' & \xrightarrow{\beta_1} & C' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow[\alpha_2]{} & B & \xrightarrow[\beta_2]{} & C & & \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ & & \operatorname{coker} f & \xrightarrow[\widetilde{\alpha_2}]{} & \operatorname{coker} g & \xrightarrow[\widetilde{\beta_2}]{} & \operatorname{coker} h & & \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

We first show that the maps $\ker f \xrightarrow{\widetilde{\alpha_1}} \ker g$, $\ker g \xrightarrow{\widetilde{\beta_1}} \ker h$, $\operatorname{coker} f \xrightarrow{\widetilde{\alpha_2}} \operatorname{coker} g$, and $\operatorname{coker} g \xrightarrow{\widetilde{\beta_2}} \operatorname{coker} h$ are well-defined, where the kernel maps are restriction of the corresponding $A' \xrightarrow{\alpha_1} B'$ and $B' \xrightarrow{\beta_1} C'$ and the cokernel maps are restriction of the corresponding $A \xrightarrow{\alpha_2} B$ and $B \xrightarrow{\beta_2} C$.

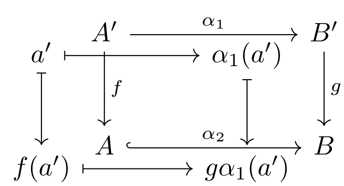
Let $a' \in \ker f$. We need to show that $\widetilde{\alpha_1}(a') \in \ker g$. As $\ker f \hookrightarrow A'$, $a' \in A'$. As $a' \in \ker f$, $f(a') = 0$, and $\alpha_2 f(a') = 0$. As the square commutes, $g\alpha_1(a') = 0$, so $\alpha_1(a') \in \ker g$, and thus $\widetilde{\alpha_1}(a') = \alpha_1(a') \in \ker g$.



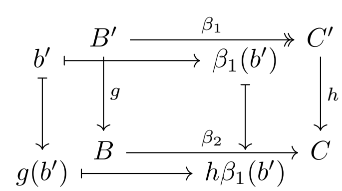
Let $b' \in \ker g$. We need to show that $\widetilde{\beta_1}(b') \in \ker h$. As $\ker g \hookrightarrow B'$, $b' \in B'$. As $b' \in \ker g$, $g(b') = 0$, and $\beta_2 g(b') = 0$. As the square commutes, $h\beta_1(b') = 0$, so $\beta_1(b') \in \ker h$, and thus $\widetilde{\beta_1}(b') = \beta_1(b') \in \ker h$.



Let $a \in \operatorname{coker} f = {}^{A}\!/_{\operatorname{im} f}$. We need to show that $\widetilde{\alpha_2}(a) \in \operatorname{coker} g = {}^{B}\!/_{\operatorname{im} g}$. In other words, we must show that $\alpha_2 : A \to B$ maps elements in the image of f to elements in the image of g. Let $a' \in A'$. Then $f(a') \in A$ is in the image of f, and α_2 maps it to $\alpha_2 f(a')$. By the commutativaty of the square, $\alpha_2 f(a') = g\alpha_1(a')$, so α_2 maps an element in the image of f to an element in the image of g, as desired.

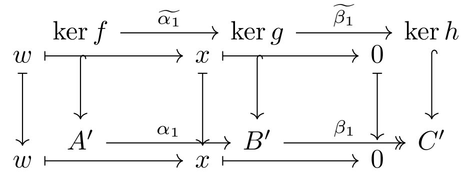


Let $b \in \operatorname{coker} g = {}^{B}\!/_{\operatorname{im} g}$. We need to show that $\widetilde{\beta_2}(b) \in \operatorname{coker} h = {}^{C}\!/_{\operatorname{im} h}$. In other words, we must show that $\beta_2 : B \to C$ maps elements in the image of g to elements in the image of h. Let $b' \in B'$. Then $g(b') \in B$ is in the image of g, and β_2 maps it to $\beta_2 g(b')$. By the commutativity of the square, $\beta_2 g(b') = h\beta_1(b')$, so β_2 maps an element in the image of g to an element in the image of h, as desired.

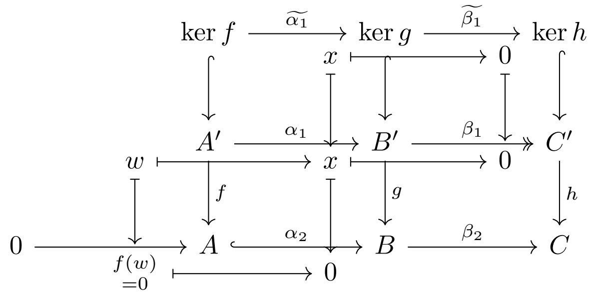


We next show exactness of $\ker f \xrightarrow{\widetilde{\alpha_1}} \ker g \xrightarrow{\widetilde{\beta_1}} \ker h \xrightarrow{\delta} \operatorname{coker} f \xrightarrow{\widetilde{\alpha_2}} \operatorname{coker} g \xrightarrow{\widetilde{\beta_2}} \operatorname{coker} h$ at all stages except δ.

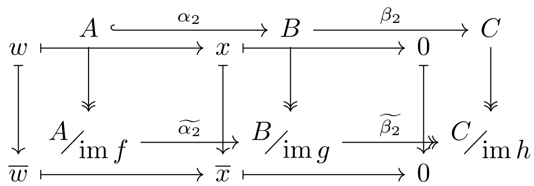
To see that $\operatorname{im} \widetilde{\alpha_1} = \ker \widetilde{\beta_1}$, we first show that $\operatorname{im} \widetilde{\alpha_1} \subseteq \ker \widetilde{\beta_1}$. Indeed, let $x \in \operatorname{im} \widetilde{\alpha_1}$. Then we compute $\widetilde{\beta_1}(x)$. Since $x \in \operatorname{im} \widetilde{\alpha_1}$, there is some $w \in \ker f$ such that $\widetilde{\alpha_1}(w) = x$. Then $\alpha_1(w) = x$ in B', and by exactness of $A' \to B' \to C'$, $\beta_1(x) = 0$. By injectivity of $\ker h \hookrightarrow C'$ and commutativity of the square, $\widetilde{\beta_1}(x) = 0$, so $x \in \ker \widetilde{\beta_1}$, as desired.



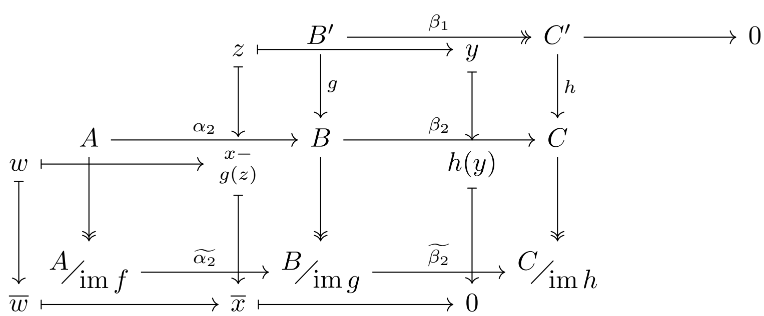
We next show that $\ker\widetilde{\beta_1} \subseteq \operatorname{im}\widetilde{\alpha_1}$. Let $x \in \ker\widetilde{\beta_1}$; then $\widetilde{\beta_1}(x) = 0$. Thus, $\beta_1(x) = 0$, and by exactness of $A' \to B' \to C'$, $x \in \operatorname{im}\alpha_1$, so there exists $w \in A'$ such that $\alpha_1(w) = x$. If $w \in \ker f$, we are done, for then $\widetilde{\alpha_1}(w) = \alpha_1(w) = x$, and $x \in \operatorname{im}\widetilde{\alpha_1}$, as desired. To see that $w \in \ker f$, see that $x \in \ker g$, so $g(x) = 0$. By commutativity of the square, $\alpha_2 f(w) = 0$. By injectivity of α_2, $f(w) = 0$, and thus $w \in \ker f$, as we needed to show.



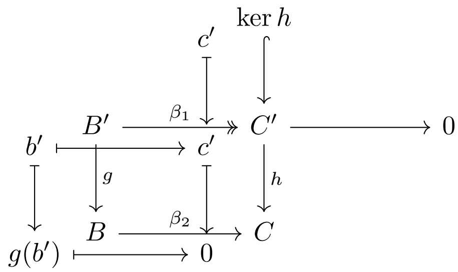
To see that $\operatorname{im}\widetilde{\alpha_2} = \ker\widetilde{\beta_2}$, we first show that $\operatorname{im}\widetilde{\alpha_2} \subseteq \ker\widetilde{\beta_2}$. Indeed, let $\overline{x} \in \operatorname{im}\widetilde{\alpha_2}$. Then we compute $\widetilde{\beta_2}(\overline{x})$. Since $\overline{x} \in \operatorname{im}\widetilde{\alpha_2}$, there is some $\overline{w} \in \operatorname{coker} f$ such that $\widetilde{\alpha_2}(\overline{w}) = \overline{x}$. Then $\alpha_2(w) = x$ in B for some representatives w and x, and by exactness of $A \to B \to C$, $\beta_2(x) = 0$. By commutativity of the square, $\widetilde{\beta_2}(\overline{x}) = 0$, so $\overline{x} \in \ker\widetilde{\beta_2}$, as desired.



We next show that $\ker\widetilde{\beta_2} \subseteq \operatorname{im}\widetilde{\alpha_2}$. Let $\overline{x} \in \ker\widetilde{\beta_2}$; then $\widetilde{\beta_2}(\overline{x}) = 0$. 0 lifts to some element $h(y) \in C$ where $y \in C'$, while $\overline{x}$ lifts to x. By surjectivity of β_1, there is some $z \in B'$ such that $\beta_1(z) = y$. By commutativity of the square, $\beta_2 g(z) = h(y)$. Also, $\beta_2(x) = h(y)$. Thus, $x - g(z) \in \ker\beta_2$, and by exactness of $A \to B \to C$, there is some $w \in A$ such that $\alpha_2(w) = x - g(z)$. Taking equivalence classes, we have $\widetilde{\alpha_2}(\overline{w}) = \overline{x - g(z)} = \overline{x}$ in $B/\operatorname{im} g$, and thus $\overline{x} \in \operatorname{im}\widetilde{\alpha_2}$, as desired.



Next, we show δ is a well-defined map. Recall that $\delta : \ker h \to \operatorname{coker} f$ is defined to be ${\alpha_2}^{-1} g {\beta_1}^{-1}$. Let $c' \in \ker h$; then $c' \in C'$. As β_1 is surjective, there exists some $b' \in B'$ such that $\beta_1(b') = c'$. Map b' to $g(b')$ in B; then by commutativity of the following square:



$\beta_2 g(b') = 0$, so $g(b') \in \ker \beta_2 = \operatorname{im} \alpha_2$. Thus there exists $a \in A$ such that $\alpha_2(a) = g(b')$. Define $\delta(c') = \overline{a}$. It remains to be seen that δ does not depend on the choice of b' or a.

Suppose b' and $\widetilde{b}'$ both are such that $\beta_1(b') = \beta_1(\widetilde{b}') = c'$. Then $\beta_1(\widetilde{b}' - b') = 0$, so $\widetilde{b}' - b' \in \ker \beta_1 = \operatorname{im} \alpha_1$, so $\alpha_1(\widetilde{a}') = \widetilde{b}' - b'$ for some $\widetilde{a}' \in A'$; i.e., $\widetilde{b}' = \alpha_1(\widetilde{a}') + b'$. Then $g(\widetilde{b}') = g(\alpha_1(\widetilde{a}') + b') = g\alpha_1(\widetilde{a}') + g(b')$. By commutativity of the square, $g\alpha_1(\widetilde{a}') = \alpha_2 f(\widetilde{a}')$, so

$$\begin{aligned} g(\widetilde{b}') &= g\alpha_1(\widetilde{a}') + g(b') \\ &= \alpha_2 f(\widetilde{a}') + \alpha_2(a) \\ &= \alpha_2(f(\widetilde{a}') + a), \end{aligned}$$

and then taking an equivalence class, $\overline{f(\widetilde{a}') + a} = \overline{a}$ in $A/\operatorname{im} f$, so δ does not depend on the choice of b'.

Suppose a and $\widetilde{a}$ are both such that $\alpha_2(a) = \alpha_2(\widetilde{a}) = g(b')$. As α_2 is injective, $a = \widetilde{a}$.

Now we show exactness at δ.

To see that $\operatorname{im} \widetilde{\beta_1} = \ker \delta$, we first show that $\operatorname{im} \widetilde{\beta_1} \subseteq \ker \delta$. Let $x \in \operatorname{im} \widetilde{\beta_1}$; i.e., there exists $w \in \ker g$ such that $\widetilde{\beta_1}(w) = x$; we need to show $\delta(x) = 0$. See that

$$\delta(x) = {\alpha_2}^{-1} g {\beta_1}^{-1}(x)$$

As δ is independent of choice, choose $w \in \ker g \subseteq B'$ to be ${\beta_1}^{-1}(x)$. Then

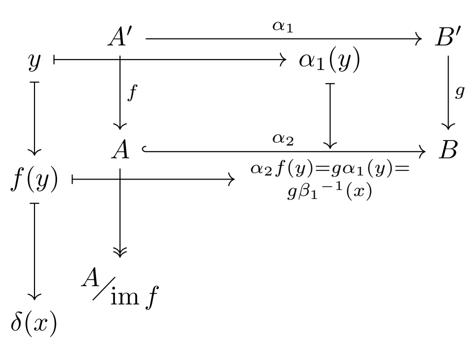
$${\alpha_2}^{-1} g {\beta_1}^{-1}(x) = {\alpha_2}^{-1} g(w) = {\alpha_2}^{-1}(0) = 0,$$

and $x \in \ker \delta$, as desired.

We next show that $\ker \delta \subseteq \operatorname{im} \widetilde{\beta_1}$. Let $x \in \ker \delta$; i.e., $\delta(x) = 0$. We need to show that there exists $w \in \ker g$ such that $\widetilde{\beta_1}(w) = x$. Since

$$\delta(x) = {\alpha_2}^{-1} g {\beta_1}^{-1}(x) = 0,$$

$\delta(x)$ lifts to an element $f(y) \in A$ for some $y \in A'$. By commutativity of the square



so ${\beta_1}^{-1}(x) - \alpha_1(y) \in \ker g$. Thus let $w = {\beta_1}^{-1}(x) - \alpha_1(y) \in \ker g$, and

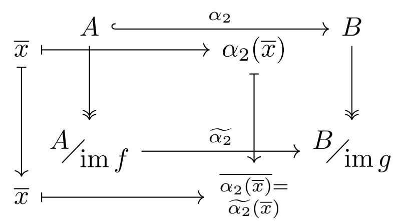
$$\widetilde{\beta_1}(w) = \widetilde{\beta_1}({\beta_1}^{-1}(x) - \alpha_1(y)) = \beta_1({\beta_1}^{-1}(x) - \alpha_1(y)) = \beta_1{\beta_1}^{-1}(x) - \beta_1\alpha_1(y) = x - 0 = x,$$

as desired.

To see that $\operatorname{im}\delta = \ker\widetilde{\alpha_2}$, we first show that $\operatorname{im}\delta \subseteq \ker\widetilde{\alpha_2}$. Let $\overline{x} \in \operatorname{im}\delta$; i.e., there exists $w \in \ker h$ such that $\delta(w) = \overline{x}$; we need to show that $\widetilde{\alpha_2}(\overline{x}) = 0$. See that

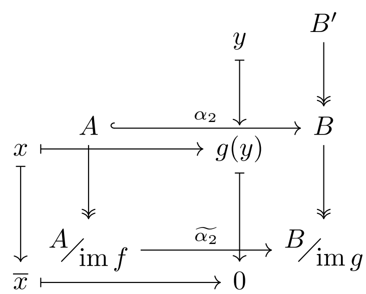
$$\begin{aligned} \delta(w) = {\alpha_2}^{-1}g{\beta_1}^{-1}(w) &= \overline{x} \\ \alpha_2{\alpha_2}^{-1}g{\beta_1}^{-1}(w) &= \alpha_2(\overline{x}) \\ g{\beta_1}^{-1}(w) &= \alpha_2(\overline{x}), \end{aligned}$$

so $\alpha_2(\overline{x})$ is in the image of g, and hence its equivalence class, equal to $\widetilde{\alpha_2}(\overline{x})$ by commutativity of the square



is 0 in $B/\operatorname{im} g$. Thus $\overline{x} \in \ker\widetilde{\alpha_2}$.

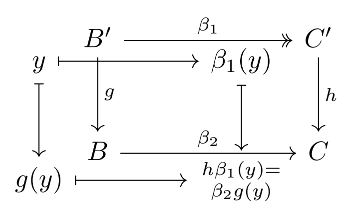
We next show that $\ker\widetilde{\alpha_2} \subseteq \operatorname{im}\delta$. Let $\overline{x} \in \ker\widetilde{\alpha_2}$; i.e., $\widetilde{\alpha_2}(\overline{x}) = 0$. We need to show that there exists $w \in \ker h$ such that $\delta(w) = \overline{x}$. As $\widetilde{\alpha_2}(\overline{x}) = 0$ in $B/\operatorname{im} g$, it lifts to an element $g(y) \in B$ where $y \in B'$, and the following square commutes:



Take $w = \beta_1(y)$. It is enough to show that $w \in \ker h$, for then

$$\delta(w) = {\alpha_2}^{-1}g{\beta_1}^{-1}(w) = {\alpha_2}^{-1}g{\beta_1}^{-1}\beta_1(y) = {\alpha_2}^{-1}g(y) = \overline{x},$$

as desired. To see that $w \in \ker h$, see that $h(w) = h\beta_1(y)$ is, by the commutativity of the square below,

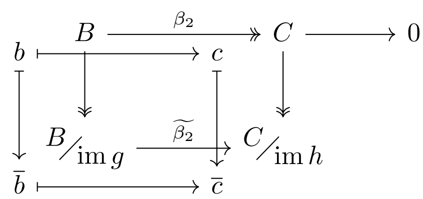


$\beta_2 g(y)$. Now, $g(y) = \alpha_2(x)$, so $\beta_2 g(y) = \beta_2\alpha_2(x) = 0$, and $w \in \ker h$, as desired.

Finally, assume $A' \to B'$ is monic and $B \to C$ is epi. We show $\ker f \to \ker g$ is monic and $\operatorname{coker} g \to \operatorname{coker} h$ is epi.

If $A' \xrightarrow{\alpha_1} B'$ is monic, then $\alpha_1(x) = 0$ implies $x = 0$. We need to show that if $\widetilde{\alpha_1}(x) = 0$, then $x = 0$. Suppose $\widetilde{\alpha_1}(x) = 0$. By definition, $\widetilde{\alpha_1}(x) = \alpha_1(x) = 0$, so $x = 0$ in B'. As $\ker g \hookrightarrow B'$ is injective, $x = 0$ in $\ker g$, as desired.

If $B \xrightarrow{\beta_2} C$ is epi, then for any $c \in C$, there exists $b \in B$ such that $\beta_2(b) = c$. We need to show that if $\overline{c} \in C/\operatorname{im} h$, then there exists $\overline{b} \in B/\operatorname{im} g$ such that $\widetilde{\beta_2}(\overline{b}) = \overline{c}$. Let $\overline{c} \in C/\operatorname{im} h$. It has some lift $c \in C$. By surjectivity of β_2, there exists $b \in B$ such that $\beta_2(b) = c$. Take the equivalence class of b, namely $\overline{b}$, and then by commutativity of



$\widetilde{\beta_2}(\overline{b}) = \overline{c}$, as desired. □